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DUALITY IN NONCONCAVE PROGRAMS USING TRANSFORMATIONS.(U)

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ABSTRACT

The canonical dual formulation (the Lagrangian function) is considered. No concavity is assumed. The existence of the nonlinear support to the hypograph of the optimum value function is proved under certain assumptions. By the transformation, this nonlinear support becomes linear and the global duality is derived.

AMS (MOS) Subject Classification: 90C30

Key Words: Optimum Value Function, Nonlinear Support, Transformation, Global Duality

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SIGNIFICANCE AND EXPLANATION

In optimization problems which possess certain geometric properties (concavity) duality relations can be obtained which are very useful in bounding or determining the extremum of one problem by the extremum of a related (dual) problem. In the present work useful duality relations are obtained in the absence of concavity.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

DUALITY IN NONCONCAVE PROGRAMS USING TRANSFORMATIONS

Okitsugu Fujiwara*

1. Introduction

We shall be concerned throughout this paper with a constrained maximization problem:

$$\text{(P)} \qquad \text{maximize } \{f(x) \text{ subject to } g(x) \geq b\}$$

where $f\colon R^n \to R^1$, $g\colon R^n \to R^m$; $f,g \in C^2$; $n \geq m$. Under the concavity assumption, the canonical dual problem which is concerned with finding a saddle point of the Lagrangian function has been extensively studied (e.g. Geoffrion [5], Rockafellar [8]). On the other hand, without the concavity assumption the Lagrangian function is no longer an appropriate function for the dual problem. Thus different types of augmented Lagrangian functions were introduced and have been intensively studied for both *local saddle points* (e.g. Arrow, Gould and Howe [1], Rockafellar [9], [10], Mangasarian [7]) and global saddle points (Rockafellar [11]).

In this paper, with no concavity assumption, we will transform the original problem so that in the transformed problem the Lagrangian function takes an adequate role for the dual problem. However, this transformed problem is by no means a concave program, hence our approach is different from the so-called concave transformability (e.g. Avriel [2], Ben-Tal [3]). We will study the hypograph

$$\hat{K} := \{(y_0,y) \in R^{m+1} \mid y_0 \leq w(y)\}$$

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of the optimum value function

$$w(y) := \max \{f(x) \text{ subject to } g(x) \geq y\} \quad .$$

Geometrically speaking, the existence of a global saddle point of the Lagrangian function can be understood to be the existence of the linear support of $\hat{K}$ at $(w(b),b)$, (Figure 1). Thus the concave program is simplified by virtue of the supporting hyperplane theorem. But in a non-concave program the supporting hyperplane theorem is no longer available (Figure 2).

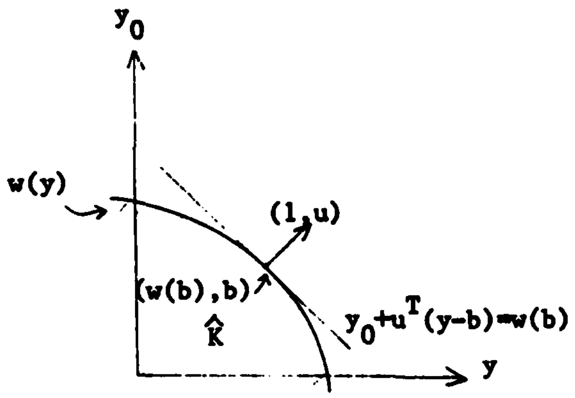

Figure 1

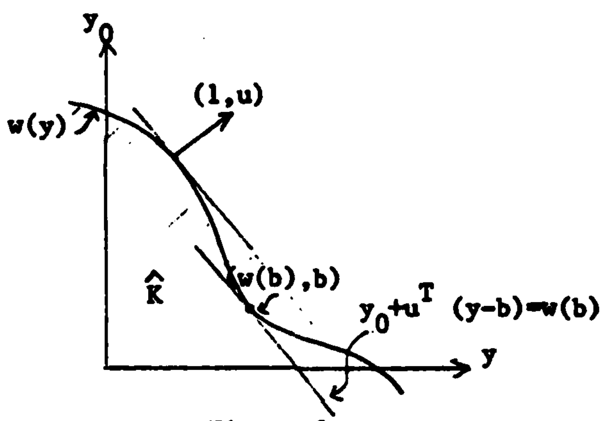

Figure 2

Our main result, under certain conditions, is to construct the nonlinear support of $\hat{K}$ at $(w(b),b)$ (Theorem 5.1). Moreover, it is shown that this non-linear support function becomes linear in the transformed space, and therefore a global saddle point of the dual problem in the transformed space gives a solution to the original problem (Theorem 7.4), (Figure 3).

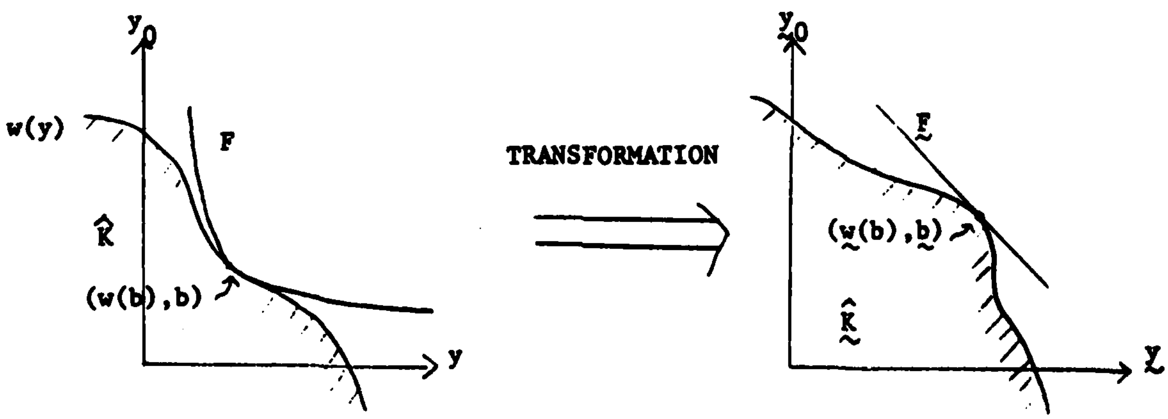

Figure 3

2. Definitions and Notation

Let us denote

$$\phi(u) := \max_{x \in R^n} \{f(x) + u^T(g(x) - b)\}$$

$$K := \{(y_o, y) \in R^{m+1} \mid (y_o, y) \leq (f(x), g(x)) \text{ for some } x \in R^n\}$$

Definition (efficient point)

$(y_o, y) \in K$ is *efficient* with respect to $S \subseteq I := \{1, 2, \ldots, m\}$, if $(z_o, z) \in K$, $(z_o, z_S) \geq (y_o, y_S) \Longrightarrow (z_o, z_S) = (y_o, y_S)$ where y_S (or z_S) is the projection of y (or z) to $R^{|S|}$.

As a particular case, $(y_o, y) \in K$ is efficient with respect to I if and only if $K \cap \{(y_o, y) + R_+^{m+1}\} = \{(y_o, y)\}$. Let $e(S)$ denote the set of all efficient points with respect to S, let $\mathrm{eff}(K) := \bigcup_{S \subseteq I} e(S)$ and let $\mathrm{eff}(X) := \{x \in R^n \mid (f(x), g(x)) \in \mathrm{eff}(K)\}$. The following statements which are easily proved explain why we are interested in efficient points.

(a) *If x^* is a unique solution of (P), then $(f(x^*), g(x^*)) \in e(I)$ and hence (P) is equivalent to*

$$\max_{\mathrm{eff}(X)} \{f(x) \text{ subject to } g(x) \geq b\}$$

(b) *If x is a solution of $\phi(u)$, then $(f(x), g(x)) \in e(S)$ where $S := \{i \mid u_i > o\}$, and hence*

$$\phi(u) = \max_{\mathrm{eff}(X)} \{f(x) + u^T(g(x) - b)\}$$

Definition (Morse Program)

(P) is called a Morse program if for any local minimum point x of (P) with $J := \{j \mid g_j(x) = b_j\}$, the following conditions are satisfied:

(CQ): $\{\nabla g_j(x) \mid j \in J\}$ are linearly independent

(SCS): there exists a unique $u \geq 0$ such that

$$\nabla f(x) + \sum_1^m u_j \nabla g_j(x) = 0$$

and

$$u_j > 0 \Longleftrightarrow j \in J.$$

(SOSC): $\mathcal{L}(x) := \nabla^2 f(x) + \sum_1^m u_j \nabla^2 g_j(x)$ is negative definite on

Kernel $\nabla g_J(x)^T$, where $g_J(x) = (g_j(x))_{j \in J}$.

Spingarn and Rockafellar [14] showed that if $f \in C^2$ and $g \in C^n$ then for almost every $(s,t) \in R^n \times R^m$

(P(s,t)): $\max \{f(x) - s^T X$ subject to $g(x) \geq b+t\}$ is a Morse program[1].

Note that this perturbation of the objective function enables us to have at most one global solution (Fujiwara [4]). Therefore under the same assumption, we can almost always expect that (P) is a Morse program with a unique solution.

Definition (total uniqueness)

(P) is totally unique if (P_S) has at most one global solution for every $S \subsetneq I$, where

---

1) A topological approach to this argument was studied by Fujiwara [4].

(P_S): $\max \{f(x) \text{ subject to } g_S(x) \geq b_S\}$

Note that if $f \in C^2$ and $g \in C^n$ then almost always (P) is expected to be totally unique, moreover (P_S) is expected to be a Morse program for all $S \subsetneq I$.

Let x^* be a local minimum point of both (P) and (P_J) where $J = \{j \mid g_j(x^*) = b_j\}$. Suppose (P) and (P_J) are Morse programs, and let u^* be the Lagrange multiplier of x^*. Then since $\begin{pmatrix} \mathcal{L}(x^*) & \nabla g_J(x^*) \\ \nabla g_J(x^*)^T & 0 \end{pmatrix}$ is nonsingular, by the implicit function theorem there exist C^1 functions $x(.)$ and $u_J(.)$ from a neighborhood $N(b_J)$ in $R^{|J|}$ to, respectively, R^n and $R^{|J|}$ such that $(x(b_J), u_J(b_J)) = (x^*, u^*)$, and for any $y_J \in N(b_J), \nabla f(x(y_J)) + \sum_J u_j(y_J)\, \nabla g_j(x(y_J)) = 0$ and $g_J(x(y_J)) = y_J$. Moreover, we have

$$\mathcal{L}(x^*)\nabla x(b_J)^T + \nabla g_J(x^*)\nabla u_J(b_J) = 0 \tag{2.1}$$

$$\nabla g_J(x^*)^T\, \nabla x(b_J)^T = I_{|J|} \tag{2.2}$$

Note that we can obtain that for any $y_J \in N(b_J)$

$$\nabla f(x(y_J)) = -\, u_J(y_J) \tag{2.3}$$

3. Nonlinear Support Functions

In this section we will define and study a parametrized family of functions which will be used as the support functions of K. Throughout this paper, a following assumption is made.

(A1): (P) is a totally unique Morse program with a unique global solution x^* and its associated Lagrange multiplier u^*.

Let $J: = \{j \mid g_j(x^*) = b_j\} = \{j \mid u_j^* > 0\}$. We will assume, in this section, that $b_i > 0$ $(i=1,\ldots, m)$ and $b_0: = f(x^*) > 0$.

Before defining the support functions, let us consider a following function[2)] for given $a_i > 0$ $(i=0,1,\ldots, m)$:

$$\text{(3.1)} \qquad \min_{0 \le i \le m} \{y_i / a_i\}$$

for $(y_0, y) \in K$.

It is easily verified that this function (3.1) has L-shaped indifference curves (Figure 4). Hence, if the maximum of (3.1) over K is attained at $(\bar{y}_0, \bar{y})$, the function (3.1) actually supports K at $(\bar{y}_0, \bar{y})$ (Figure 5).

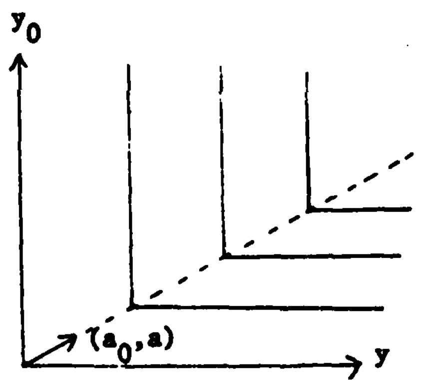

Figure 4

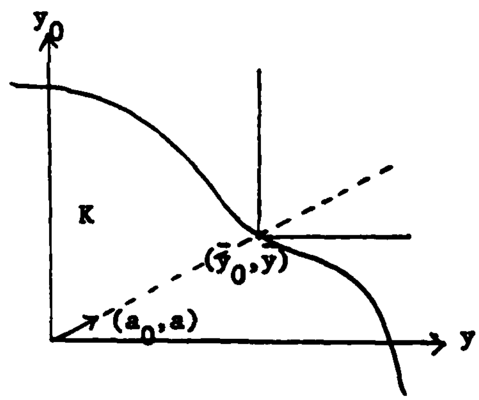

Figure 5

This fact leads us to consider the so-called mean value functions studied by Hardy, Littlewood and Polya [6], because they showed

2) First used by Scarf [12] in the analysis of discrete programming problems.

Lemma 3.2 ([6], Theorems 3 & 4)

Let $y_i > 0$, $a_i > 0$ $(i=0,1,\ldots, m)$, $\sum_0^m a_i = 1$. Then we have

$$(\sum_0^m a_i\, y_i^q)^{1/q} \downarrow \min_i \{y_i\} \qquad \text{as} \quad q \to -\infty \tag{3.3}$$

$$(\sum_0^m a_i\, y_i^q)^{1/q} \uparrow \max_i \{y_i\} \qquad \text{as} \quad q \to +\infty \tag{3.4}$$

$$(\sum_0^m a_i\, y_i^q)^{1/q} \to \prod_0^m y_i^{a_i} \qquad \text{as} \quad q \to 0 \tag{3.5}$$

Thus we now define a parametrized family of functions $\{F_q\}_{q\geq -1}$ defined on R_{++}^{m+1} as follows:

$$F_q(y_0,y) := (\sum_0^m c_i\, y_i^{-q})^{-1/q}$$

for $(y_0,y) \in R_{++}^{m+1}$, $q \geq -1$ $(q \neq 0)$; and

$$F_0(y_0,y) := \lim_{q\to 0} F_q(y_0,y)$$

where $\quad c_i = u_i^* b_i^{q+1} \Big/ \sum_0^m u_j^*\, b_j^{q+1} \quad (i=0,1,\ldots, m),\ u_0^* = 1.$

Note that $\sum_0^m c_i = 1$ and the above limit exists by [6] Theorem 3. It is shown that the functions F_q are concave ([6] Theorem 24) and especially F_{-1} is a linear function.

Then by Lemma 3.2, we have

Proposition 3.6

<u>We have that</u>

$$F_q(y_o,y) \to \max_{J_o}\{b_j\}\cdot\min_{J_o}\{y_i/b_i\} \quad \text{as } q\to\infty \tag{3.7}$$

$$F_q(y_o,y) \to \prod_{J_o} b_j^{c_j^o} \prod_{J_o}(y_i/b_i)^{c_j^o} \quad \text{as } q\to 0 \tag{3.8}$$

<u>where</u> $J_o := J\cup\{0\}$ $\quad c_i^o := u_i^* b_i \Big/ \sum_0^m u_j^* b_j \quad (i=0,1,\ldots,m)$.

<u>Moreover, the convergence</u> (3.7) <u>is uniform on any compact set of</u> R_{++}^{m+1}.

<u>proof</u>

$$F_q(y_o,y) = \{\sum_0^m (u_i^* b_i^{q+1} \Big/ \sum_0^m u_j^* b_j^{q+1})\, y_i^{-q}\}^{-1/q}$$

$$= \{\sum_0^m (u_j^* b_j) b_j^q\}^{1/q} \{\sum_0^m (u_i^* b_i)(y_i/b_i)^{-q}\}^{-1/q}$$

$$= \{\sum_0^m (u_j^* b_j \Big/ \sum_0^m u_k^* b_k)\, b_j^q\}^{1/q} \cdot \{\sum_0^m (u_i^* b_i \Big/ \sum_0^m u_k^* b_k)(y_i/b_i)^{-q}\}^{-1/q}$$

The first term $\uparrow \max_{J_o}\{b_j\}$ as $q\to\infty$ by (3.4), and the second term $\downarrow \min_{J_o}\{y_i/b_i\}$ as $q\to\infty$ by (3.3).

By Dini's Theorem[3)], the second convergence is uniform on any compact set. Therefore, by the next lemma which is easily verified, the convergence

---

3) <u>Dini's Theorem</u> If a sequence of continuous functions, defined on a compact metric space, converges pointwise to a continuous function monotonically, then the convergence is uniform.

(3.7) is uniform on any compact set of R^{m+1}_{++}. We also have that by (3.5) the first term $\to \underset{J_o}{\pi}\, b_j^{c_j^o}$ as $q \to 0$ and the second term $\underset{J_o}{\pi}\, (y_i/b_i)^{c_i^o}$ as $q \to 0$.

Hence, (3.8) is obtained

<u>QED</u>

<u>Lemma</u> 3.9 <u>Let</u> $\{h_n\}$, h <u>be real valued continuous functions on a compact set</u> D <u>in</u> R^γ <u>for some</u> $\gamma \geq 1$. <u>Let</u> $\{a_n\}$, a <u>be real numbers such that</u> $a_n \to a$. If h_n <u>converges uniformly to</u> h, <u>then</u> $a_n h_n$ <u>converges uniformly to</u> ah <u>on</u> D.

<u>Remark</u> By Proposition 3.6 the behavior of the indifference curves evaluated at (y_o^*, y^*), according to the increase of q from -1 to $+\infty$, is illustrated in Figure 6.

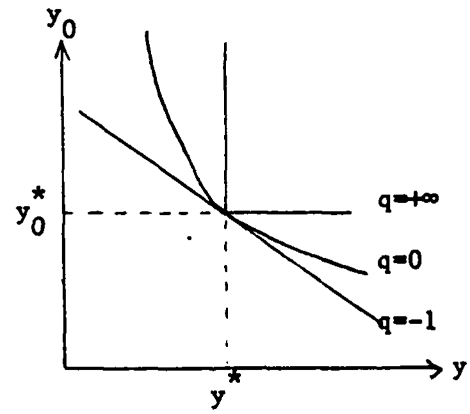

Figure 6

4. <u>Local Support Theorem</u>

In this section, under the certain assumptions, we will show that the parametrized family of functions $\{F_q\}_{q\geq -1}$ locally supports K at (w(b),b) for sufficiently large q. Throughout this section, in addition to the assumption (A1), the following assumptions are made:

(A2): $(y_o^*, y^*) := (f(x^*), b)$ is efficient with respect to

$$J := \{i \mid g_i(x^*) = b_i\}$$

(A3): eff(K) is compact

(A4): $w_J(y_J) = f(x(y_J))$ in a neighborhood of b_J, where $w_J(y_J) := \max\{f(x) \text{ subject to } g_J(x) \geq y_J\}$ and $x(.)$ is the implicit function defined at the end of section 2.

We will prove the following result,

<u>Theorem</u> 4.1 <u>(Local Support)</u>

Under the assumptions (A1) ~ (A4), a program

$$(4.2) \qquad \max_{R_{++}^{|J_o|}} \{F_q(y_o, y_J) \text{ subject to } w_J(y_J) \geq y_o\}$$

<u>attains a strict local maximum at</u> (y_o^*, y_J^*) <u>for sufficiently large</u> $q > -1$, <u>where</u> $F_q(y_o, y_J) = (\sum_{J_o} c_i y_i^{-q})^{-1/q}$ and $J_o := J \cup \{0\}$. <u>In other words, there exists</u> $q_o > -1$ <u>and for any</u> $q \geq q_o$ <u>there exists an open neighborhood</u> U_q of (y_o^*, y_J^*) <u>in</u> $R_{++}^{|J_o|}$ <u>such that for any</u> $(y_o, y_J) \in U_q \cap \{(y_o, y_J) \in R^{|J_o|} \mid w_J(y_J) \geq y_o\}$, <u>we have</u>

$$F_q(y_o, y_J) \leq F_q(y_o^*, y_J^*)$$

<u>where equality holds only if</u> $(y_o, y_J) = (y_o^*, y_J^*)$.

To prove this theorem we need some preliminary results and we start to prove the following lemma.

Lemma 4.3

If (y_o^*, y^*) is efficient with respect to J, then x^* is also a unique global solution of (P_J).

proof

It suffices to show that if $f(x) \geq f(x^*)$ and $g_J(x) \geq b_J$, then $x = x^*$. Since $(f(x), g_J(x)) \geq (f(x^*), b_J) = (y_o^*, y_J^*)$ and $(y_o^*, y^*) \in e(J)$, we have that $(f(x), g_J(x)) = (y_o^*, y_J^*)$. Since (P_J) is totally unique, a solution x of (P_J) must be x^*.

QED

Note that we have obtained $w_J(b_J) = w(b)$.

For computational convenience, instead of (4.2) we consider a program

$$(4.4) \qquad \max_{R_{++}^{|J_o|}} \{G_q(y_o, y_J) \text{ subject to } w_J(y_J) \geq y_o\}$$

where $G_q(y_o, y_J) = \log F_q(y_o, y_J)$; and we assume that $J = \{1, 2, \ldots, \ell\}$. Since eff(K) is compact by (A3), we can translate all coordinates so that we have that all $y_i > 0$ $(i=0,1,\ldots, m)$ in a neighborhood of eff(K).

Proposition 4.5

In a program (4.4), (y_o^*, y_J^*) satisfies the Kuhn-Tucker condition with the Lagrange multiplier $v^* = (u^{*T}b)^{-1}$ which does not depend on q.

<u>proof</u> Let $L(y_o, y_J, v)$ be the Lagrangian function of (4.4). Then by (A4) and (2.3), $\nabla L(y_o, y_J, v) = \nabla G_q(y_o, y_J) + v\nabla(w_J(y_J) - y_o)$, namely $\frac{\partial L}{\partial y_j} = c_j y_j^{-q-1} \cdot (\sum_o^m c_i y_i^{-q})^{-1} - vu_j(y_J)$ for $j=0,1,\dots,\ell$ (recall that $u_o = 1$). Hence the Kuhn-Tucker condition becomes

$$vu_j(y_J) = c_j y_j^{-q-1} \cdot (\sum_o^m c_i y_i^{-q})^{-1} \qquad j=0,1,\dots,\ell$$

Solving these equations at (y_o^*, y_J^*), we obtain

$$v^* = (u^{*T} b)^{-1}$$

<u>QED</u>

F_q is a concave function and log(.) is a strictly increasing concave function, hence G_q is a concave function for $q \geq -1$. Then the Hessian of G_q, $\nabla^2 G_q$ is negative semidefinite on R^n. Restricting $\nabla^2 G_q$ on a subspace we have,

<u>Lemma</u> 4.6

$\nabla^2 G_q(y_o^*, y_J^*)$ <u>is negative definite on</u> Ker $\nabla(w_J(y_J^*) - y_o^*)^T$ <u>for any</u> $q > -1$.

<u>proof</u>

Note that we have

$$\frac{\partial^2 G_q}{\partial y_i \partial y_j} = \begin{cases} q \left(\frac{\partial G_q}{\partial y_i}\right)\left(\frac{\partial G_q}{\partial y_j}\right) & \text{if } i \neq j \\ -(q+1)\, y_i^{-1} \left(\frac{\partial G_q}{\partial y_i}\right) + q \left(\frac{\partial G_q}{\partial y_i}\right)^2 & \text{if } i = j \end{cases}$$

and hence

$$\frac{\partial^2 G_q(y_o^*, y_J^*)}{\partial y_i \partial y_j} = \begin{cases} q v^{*2} u_i^* u_j^* & \text{if } i \neq j \\ -(q+1) b_i^{-1} v^* u_i^* + q v^* u_i^* & \text{if } i = j \end{cases}$$

Since $\nabla(w_J(y_J^*) - y_o^*) = -(1, u_J^*)$, we have that $s \neq 0 \in \text{Ker } \nabla(w_J(y_J^*) - y_o^*)^T$ if and only if $s_o = -\sum_1^{\ell} u_j^* s_j$. This implies that $s \neq 0 \iff s_J \neq 0$ because $u_j^* > 0$ for $j=1,\ldots,\ell$. Thus we have that

$$s^T \nabla^2 G_q(y_o^*, y_J^*) s = \sum_{i,j} s_i s_j \frac{\partial^2 G_q}{\partial y_i \partial y_j}$$

$$= 2 \sum_{j \geq 1} s_o s_j \frac{\partial^2 G_q}{\partial y_o \partial y_j} + \sum_{\substack{j \neq k \\ j,k \geq 1}} s_j s_k \frac{\partial^2 G_q}{\partial y_j \partial y_k}$$

$$+ s_o^2 \frac{\partial^2 G_q}{\partial y_o^2} + \sum_{j \geq 1} s_j^2 \frac{\partial^2 G_q}{\partial y_j^2}$$

$$= (*)$$

First term $= 2\{ \sum_{j \geq 1} (- \sum_{k \geq 1} u_k^* s_k) s_j q v^{*2} u_j^* \} = -2 q v^{*2} (\sum_{j \geq 1} u_j^* s_j)^2$

Second term $= q v^{*2} \sum_{\substack{j \neq k \\ j,k \geq 1}} u_j^* u_k^* s_j s_k$

Third term $= q v^{*2} (\sum_{j \geq 1} u_j^* s_j)^2 - (q+1) b_o^{-1} v^* (\sum_{j \geq 1} u_j^* s_j)^2$

Fourth term $= q v^{*2} \sum_{j \geq 1} (u_j^* s_j)^2 - (q+1) v^* (\sum_{j \geq 1} b_j^{-1} u_j^* s_j^2)$

Hence $\quad (*) = -(q+1) v^* \{ b_o^{-1} (\sum_{j \geq 1} u_j^* s_j)^2 + \sum_{j \geq 1} b_j^{-1} u_j^* s_j^2 \} < 0$

for $s_J \neq 0$ and for $q > -1$.

<u>QED</u>

By (2.3) and by the assumption (A4), we have

$$\nabla^2 L(y_o^*, y_J^*, v^*) = \nabla^2 G_q(y_o^*, y_J^*) - v^* \begin{pmatrix} 0 & 0 \\ 0 & \nabla u_J(y_J^*) \end{pmatrix}$$

<u>Proposition</u> 4.7

<u>Let</u> $A(q) := \nabla^2 G_q(y_o^*, y_J^*)$ <u>and let</u> $B := -v^* \begin{pmatrix} 0 & 0 \\ 0 & \nabla u_J(y_J^*) \end{pmatrix}$. <u>Then</u> $A(q) + B$ is negative definite on Ker $\nabla(w_J(y_J^*) - y_o^*)^T$ for sufficiently large $q > -1$.

<u>Remark</u> 4.8 If $\nabla u_J(y_J^*)$ is positive definite on $R^{|J|}$, then B is negative definite on Ker $\nabla(w_J(y_J^*) - y_o^*)^T$. Hence $A(q) + B$ is negative definite on Ker $\nabla(w_J(y_J^*) - y_o^*)^T$ for all $q \geq -1$, because G_q is concave and so $A(q)$ is negative semidefinite on $R^{|J|+1}$.

By the elementary computation of matrices we obtain,

<u>Lemma</u> 4.9

<u>Let</u> $$E := \begin{pmatrix} -u_1^* & -u_2^* & \cdots & -u_\ell^* \\ 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & & & 1 \end{pmatrix}$$ <u>and let</u>

$$D := \begin{pmatrix} b_o^{-1} u_1^{*2} + b_1^{-1} u_1^* & b_o^{-1} u_1^* u_2^* & \cdots\cdots & b_o^{-1} u_1^* u_\ell^* \\ b_o^{-1} u_2^* u_1^* & b_o^{-1} u_2^{*2} + b_2^{-1} u_2^* & & \vdots \\ \vdots & & \ddots & \\ b_o^{-1} u_\ell^* u_1^* & \cdots\cdots & & b_o^{-1} u_\ell^{*2} + b_\ell^{-1} u_\ell^* \end{pmatrix}.$$

Then we have

$$E^T A(q) E = -(q+1) v^* D \quad .$$

Note that $\quad \mathrm{Ker}\ \nabla(w_J(y_J^*) - y_o^*)^T = \mathrm{Im}\ E$.

Lemma 4.10

Let $d_1, \ldots, d_\ell$ be eigenvalues of D, then we have that all $d_i > 0$ $(i=1,\ldots,\ell)$ and $-(q+1)v^* d_1, \ldots, -(q+1)v^* d_\ell$ are eigenvalues of $E^T A(q) E$.

proof

A(q) is negative definite on Im E (by Lemma 4.6)

$\Longleftrightarrow$ $E^T A(q) E$ is negative definite on R^ℓ

$\Longleftrightarrow$ D is positive definite on R^ℓ (by Lemma 4.9)

The last assertion is obvious by Lemma 4.9.

QED

Let P be an orthogonal matrix such that

$$P^T (E^T A(q) E) P = \begin{pmatrix} -(q+1)v^* d_1 & & 0 \\ & \ddots & \\ 0 & & -(q+1)v^* d_\ell \end{pmatrix} . \tag{4.11}$$

Then we have

Lemma 4.12

$P^T E^T (A(q) + B) E P$ is negative definite on R^ℓ for sufficiently large $q > -1$.

proof

Let $A = (a_{ij}) = P^T E^T B E P$, then by (4.10) we have

$$P^T E^T (A(q)+B) E P = \begin{pmatrix} -(q+1)v^* d_1 + a_{11} & a_{12} & \cdots & a_{1\ell} \\ a_{21} & & & \\ \vdots & & & \\ a_{\ell 1} & & & -(q+1)v^* d_\ell + a_{\ell\ell} \end{pmatrix}.$$

Pick up a sufficiently large $\bar{q}$ that satisfies

$$(4.13) \qquad -(\bar{q}+1)v^* d_i + a_{ii} < 0 \qquad \text{for } i=1,\ldots,\ell$$

and

$$(4.14) \qquad \min_i \left| -(\bar{q}+1)v^* d_i + a_{ii} \right| > \max_{j\neq k} |a_{jk}| \cdot (\ell^2-\ell)$$

Thus for any $s \neq 0 \in R^\ell$ with $|s_r| = \max_i |s_i|$ and for any $q \geq \bar{q}$, we have that

$$\begin{aligned} s^T P^T E^T (A(q)+B) E P s &= \sum_1^\ell \{-(q+1)v^* d_i + a_{ii}\} s_i^2 + \sum_{j\neq k} a_{jk} s_j s_k \\ &\leq \{-(q+1)v^* d_r + a_{rr}\} s_r^2 + \max_{j\neq k} |a_{jk}| \cdot (\ell^2-\ell) s_r^2 < 0 \end{aligned}$$

by (4.13) and (4.14).

QED

Then Proposition 4.7 follows immediately. By Propositions 4.5 and 4.7, Theorem 4.1 follows since (y_o^*, y_J^*) satisfies the second order sufficiency conditions for the local optimality of (4.4).

5. Global Support Theorem

In this section, we will show that $\{F_q\}_{q\geq -1}$ supports eff(K) at (y_o^*,y^*) for sufficiently large q. The following assumption is made in this section:

(AX): In Theorem 4.1, we assume $U = U_q$ for all $q \geq q_o$.

Then the global support theorem will be

Theorem 5.1 (Global Support)

Under the assumptions (A1) ~ (A4) *and* (AX), *for sufficiently large* $q > -1$ *and for any* $(y_o,y) \in \text{eff}(K)$, *we have* $F_q(y_o,y) \leq F_q(y_o^*,y^*)$ *where equality holds only if* $(y_o,y_J) = (y_o^*,y_J^*)$.

proof We separate this proof into two parts[4]: a neighborhood V of (y_o^*,y^*) and outside this neighborhood, eff(K) − V .

(1) V Let $V: = U\times R_{++}^{|I\setminus J_o|} \cap \{(y_o,y)\,|\,w(y) \geq y_o\}$.

$(y_o',y') \in V \Longrightarrow w_J(y_J') \geq w(y') \geq y_o' \Longrightarrow (y_o',y_J') \in U \cap \{(y_o,y_J)\,|$

$w_J(y_J) \geq y_o\} \Longrightarrow F_q(y_o',y_J') \leq F_q(y_o^*,y_J^*)$ by Theorem 4.1 $\Longrightarrow F_q(y_o',y')$ $\leq F_q(y_o^*,y^*)$ and equality holds only if $(y_o',y_J') = (y_o^*,y_J^*)$. Since V does not depend on $q \geq q_o$, F_q supports eff(K) $\cap$ V at (y_o^*,y^*) for $q \geq q_o$.

(2) $\text{eff}(K) - U\times R_{++}^{|I\setminus J_o|} = \text{eff}(K) - V$

---

4) This idea of resolving the global support theorem into a local and nonlocal components, when the set in question is compact, is due to Westhoff [15].

Since $(y_o^*, y^*) \in e(J)$, $\min_{J_o}\{y_i/b_i\} \leq \min_{J_o}\{y_i^*/b_i\} = 1$ on K and equality holds only if $(y_o, y_J) = (y_o^*, y_J^*)$. Because $\min_{J_o}\{y_i/b_i\} \geq 1 \Longrightarrow y_i \geq b_i = y_i^*$ for all $i \in J_o \Longrightarrow (y_o, y_J) = (y_o^*, y_J^*)$.

Since $\{(y_o, y) \in K \mid (y_o, y_J) = (y_o^*, y_J^*)\} \cap R_{++}^{m+1} \subseteq V$, we have that

(5.2) $$\min_{J_o}\{y_i/b_i\} < \min_{J_o}\{y_i^*/b_i\}$$

on a compact set eff(K) - V .

Let $h_q(y_o, y) := F_q(y_o, y) - F_q(y_o^*, y^*)$ and $h(y_o, y) := \max_{J_o}\{b_j\} \cdot (\min_{J_o}\{y_i/b_i\} - \min_{J_o}\{y_i^*/b_i\})$. Then $h_q \rightarrow h$ uniformly on eff(K) - V by Proposition 3.6, and we have that $h < 0$ on eff(K) - V by (5.2). Therefore, by the following lemma, we have that $h_q < 0$ on eff(K) - V for sufficiently large q.

<u>QED</u>

<u>Lemma</u> 5.3 <u>Let</u> $\{h_q\}$ <u>and</u> h <u>be real-valued continuous functions on a compact set</u> D <u>in</u> R^k <u>for some</u> $k \geq 1$. <u>If</u> h_q <u>converges uniformly to</u> h <u>and if</u> $h(x) < 0$ <u>for all</u> $x \in D$, <u>then there exists</u> q_1 <u>such that for any</u> $q \geq q_1$ <u>and for any</u> $x \in D$, <u>we have</u> $h_q(x) < 0$.

<u>proof</u> Suppose not, then for any $k \geq 1$, there exists $q_k \geq k$ and $x_k \in D$ such that $h_{q_k}(x_k) \geq 0$. Since D is compact, there exists a converging subsequence of $\{x_k\}$. For notational convenience let us assume $x_k \rightarrow x_o \in D$. By the continuity of h at x_o and by the uniform convergence of $\{h_q\}$, we have that for any $\varepsilon > 0$ there exists $n(\varepsilon)$ such that for any $k \geq n(\varepsilon)$ $|h(x_k) - h(x_o)| < \frac{\varepsilon}{2}$ and $|h_{q_k}(x_k) - h(x_k)| < \frac{\varepsilon}{2}$ hold. Then for $k \geq n(\varepsilon)$, $|h_{q_k}(x_k) - h(x_o)| \leq$

$|h_{q_k}(x_k) - h(x_k)| + |h(x_k) - h(x_o)| < \varepsilon$. This implies that $h_{q_k}(x_k) < 0$ for sufficiently large k, which contradicts the choice of $\{q_k\}$ and $\{x_k\}$.

QED

6. Transformation

In this section we define a transformation which is coordinate independent and strictly increasing. It is shown that by this transformation the functions $\{F_q\}_{q>0}$ are transformed to be linear.

For $q > 0$ and $y_i > 0$ ($i=0,1,\ldots, m$), we define a transformation used by Scarf [13] by

$$\underset{\sim}{y}_i := 1 - y_i^{-q}$$

for $i=0,1,\ldots, m$. We denote the transformed spaces of eff(K) and R^{m+1}_{++} by $\underset{\sim}{\mathrm{eff}}(K)$ and $\underset{\sim}{R}^{m+1}_{++}$. Let us define functions $\{\underset{\sim}{F}_q\}_{q>0}$ defined on $\underset{\sim}{R}^{m+1}_{++}$ by

$$\underset{\sim}{F}_q(\underset{\sim}{y}_o,\underset{\sim}{y}) := 1 - F_q(y_o,y)^{-q}$$

Then we have that

$$\begin{aligned}
\underset{\sim}{F}_q(\underset{\sim}{y}_o,\underset{\sim}{y}) &= 1 - \left[\left(\sum_o^m c_i y_i^{-q}\right)^{-1/q} \right]^{-q} \\
&= 1 - \sum_o^m c_i y_i^{-q} \\
&= \sum_o^m c_i (1 - y_i^{-q}) \qquad \text{since } \sum_o^m c_i = 1 \\
&= \sum_o^m c_i \underset{\sim}{y}_i
\end{aligned}$$

This shows that for any $q > 0$, a nonlinear function F_q is transformed to a linear function $\underset{\sim}{F}_q$. Since this transformation is strictly increasing for $q > 0$, we have that F_q supports eff(K) at (y_o^*, y^*) if and only if $\underset{\sim}{F}_q$ supports eff($\underset{\sim}{K}$) at $(\underset{\sim}{y}_o^*, \underset{\sim}{y}^*)$ (see Figure 3).

7. Global Duality Theorem

In this section we prove the strong duality theorem of the canonical dual formulation in the transformed space. First of all a sufficient condition for the strong duality theorem is discussed.

Lemma 7.1

If x_{u*} *is a solution of* $\phi(u^*)$ *satisfying*

$$g(x_{u*}) \geq b \tag{7.2}$$

$$u^{*T}(g(x_{u*}) - b) = 0 \tag{7.3}$$

then $\phi(u^*) = w(b) = f(x_{u*})$.

proof Let x_v be a solution of $\phi(v)$ for $v \geq 0$. Then we have that

$$\begin{aligned}
\phi(v) - \phi(u^*) &= f(x_v) + v^T(g(x_v) - b) - f(x_{u*}) - u^{*T}(g(x_{u*}) - b) \\
&\geq f(x_{u*}) + v^T(g(x_{u*}) - b) - f(x_{u*}) - u^{*T}(g(x_{u*}) - b) \\
&= (v - u^*)^T(g(x_{u*}) - b) \\
&= v^T(g(x_{u*}) - b) && \text{by (7.2)} \\
&\geq 0 && \text{by (7.1) and } v \geq 0.
\end{aligned}$$

If $\phi(v)$ has no solution for some $v \geq 0$, then there exists $\bar{x}_v$ such that

$$\phi(v) \geq f(\bar{x}_v) + v^T(g(\bar{x}_v) - b) \geq f(x_{u^*}) + v^T(g(x_{u^*}) - b).$$

Hence in this case we still have $\phi(v) \geq \phi(u^*)$. Therefore we obtain

$$\phi(u^*) = \min_{v \geq 0} \phi(v).$$

For any $x \in R^n$ satisfying $g(x) \geq b$, we have that

$$\begin{aligned} f(x_{u^*}) &= f(x_{u^*}) + u^{*T}(g(x_{u^*}) - b) && \text{by (7.3)} \\ &= \phi(u^*) \\ &\geq f(x) + u^{*T}(g(x) - b) \\ &\geq f(x) && \text{since } u^* \geq 0,\ g(x) \geq b. \end{aligned}$$

Therefore $w(b) = f(x_{u^*}) = \phi(u^*)$

<u>QED</u>

Now we state the main theorem,

<u>Theorem</u> 7.4 (Global Duality)

<u>Under the assumptions</u> (A1) ~ (A4) <u>and</u> (AX), <u>we have</u>

$$\underset{\sim}{f}(x^*) = \min_{u \geq 0} \max_{\mathrm{eff}(X)} \{\underset{\sim}{f}(x) + u^T(\underset{\sim}{g}(x) - \underset{\sim}{b})\}$$

<u>for sufficiently large</u> q, <u>where</u> $\underset{\sim}{f}(x) = 1 - f(x)^{-q}$, $\underset{\sim}{g}_i(x) = 1 - g_i(x)^{-q}$, $\underset{\sim}{b}_i = 1 - b_i^{-q}$ for $x \in \mathrm{eff}(X)$ and for $i=1,\ldots, m$.

<u>proof</u>

$$F_q(y_o^*, y^*) \geq F_q(y_o, y) \qquad \text{on eff}(K)$$

$$\Longleftrightarrow \quad \underset{\sim}{F}_q(\underset{\sim}{y}_o^*, \underset{\sim}{y}^*) \geq \underset{\sim}{F}_q(\underset{\sim}{y}_o, \underset{\sim}{y}) \qquad \text{on eff}(\underset{\sim}{K})$$

$$\Longleftrightarrow \quad \sum_o^m c_i \underset{\sim}{y}_i^* \geq \sum_o^m c_i \underset{\sim}{y}_i \qquad \text{on eff}(\underset{\sim}{K})$$

$$\Longleftrightarrow \quad \sum_o^m u_i^* b_i^{q+1} \underset{\sim}{y}_i^* \geq \sum_o^m u_i^* b_i^{q+1} \underset{\sim}{y}_i \qquad \text{on eff}(\underset{\sim}{K})$$

$$\Longrightarrow \quad b_o^{q+1} \underset{\sim}{f}(x^*) + \sum_1^m u_i^* b_i^{q+1} \underset{\sim}{g}_i(x^*) \geq b_o^{q+1} \underset{\sim}{f}(x) + \sum u_i^* b_i^{q+1} \underset{\sim}{g}_i(x) \qquad \text{on eff}(X)$$

$$\Longleftrightarrow \quad \underset{\sim}{f}(x^*) \geq \underset{\sim}{f}(x) + \sum_1^m u_i^* (b_i/b_o)^{q+1} (\underset{\sim}{g}_i(x) - \underset{\sim}{b}_i) \qquad \text{on eff}(X)$$

Let $\hat{\phi}(u) = \max\limits_{\text{eff}(X)} \{\underset{\sim}{f}(x) + u^T(\underset{\sim}{g}(x) - \underset{\sim}{b})\}$ for $u \geq 0$. Then $\underset{\sim}{f}(x^*) = \hat{\phi}(v^*)$ with $v_i^* = u_i^*(b_i/b_o)^{q+1}$ for $i=1,\ldots,m$. However since $\underset{\sim}{g}(x^*) \geq \underset{\sim}{b}$ and $v^{*T}(\underset{\sim}{g}(x^*) - \underset{\sim}{b}) = 0$ hold, by Lemma 7.1 we have that $\hat{\phi}(v^*) = \min\limits_{u \geq 0} \hat{\phi}(u)$. Hence, by Theorem 5.1 we complete the proof.

<u>QED</u>

8. <u>A Sufficient Condition for Assumption</u> (AX)

In this section (A1) ~ (A4) are assumed, and we will show that the strict local concavity at x^* is a sufficient condition for the assumption (AX).

<u>Definition</u> A program (P) is locally concave at a local minimum point x, if $\mathcal{X}(x) = \nabla^2 f(x) + \sum_1^m u_j \nabla^2 g_j(X)$ is negative semidefinite on R^n where u is the associated Lagrange multiplier of x. (P) is <u>locally strictly concave</u> at x if $\mathcal{X}(x)$ is negative definite on R^n.

Lemma 8.1

If $\mathcal{L}(x^*)$ *is negative (semi)definite on* R^n, *then* $\nabla u_J(b_J)$ *is positive (semi)definite on* $R^{|J|}$.

proof For any $s \in R^{|J|}$, we have

$$s^T \nabla u_J(b_J)\, s = (s^T \nabla x(b_J) \nabla g_J(x^*))\ \nabla u_J(b_J) s \qquad \text{by (2.2)}$$

$$= s^T \nabla x(b_J)(-\mathcal{L}(x^*)\ \nabla x(b_J)^T) s \qquad \text{by (2.1)}$$

$$= -(\nabla x(b_J)^T s)^T\, \mathcal{L}(x^*)(\nabla x(b_J)^T s)$$

Hence $\nabla u_J(b_J)$ is positive (semi)definite on $R^{|J|}$ if and only if $\mathcal{L}(x^*)$ is negative (semi)definite on Im $\nabla x(b_J)^T$; because by (2.2) $\nabla x(b_J)^T$ has full rank. Since $\mathcal{L}(x^*)$ is negative (semi)definite on R^n, it follows that $\nabla u_J(b_J)$ is positive (semi)definite on $R^{|J|}$.

QED

By the assumption (A4) and by (2.3), we have $\nabla^2 w_J(b_J) = -\nabla u_J(b_J)$. Hence Lemma 8.1 implies that $w_J(y_J)$ is strictly concave in a neighborhood of b_J if (P) is locally strictly concave at x^*. So we obtain

Proposition 8.2 *Assume* (A1) ~ (A4) *are satisfied.*

If (P) *is locally strictly concave at* x^*, *then the assumption* (AX) *holds*

proof

Let N be a convex neighborhood of b_J in $R_{++}^{|J|}$ such that $w_J(.)$ is strictly concave on N by Lemma 8.1. Then $M := \{(y_o, y_J) \in R_{++}^{|J_0|} \mid w_J(y_J) \geq y_o,\ y_J \in N\}$

is a convex set. Let $U: = R^1_{++} \times N$. Then $U \cap \{(y_o,y_J) \in R^{|J_o|}_{++} \mid w_J(y_J) \geq y_o\} = M$. By remark 4.8, for any $q \geq -1$, (y^*_o,y^*_J) attains a strict local maximum of F_q on $\{(y_o,y_J) \in R^{|J_o|}_{++} \mid w_J(y_J) \geq y_o\}$. Hence, it is also a strict local maximum of F_q on the smaller domain M. However, since F_q is a strict concave function and since M is a convex set, (y^*_o,y^*_J) is a unique global maximum of F_q on M. Namely, we have shown that for the neighborhood U of (y^*_o,y^*_J) in $R^{|J_o|}_{++}$ we have that for any $q \geq -1$ and for any $(y_o,y_J) \in U \cap \{(y_o,y_J) \in R^{|J_o|} \mid w_J(y_J) \geq y_o\}$,

$$F_q(y_o,y_J) \leq F_q(y^*_o,y^*_J)$$

holds where equality holds only if $(y_o,y_J) = (y^*_o,y^*_J)$

QED

9. Optimum Value Function with A Quadratic Term

In this last section a parametrized quadratic term is considered. We subtract a quadratic term from the optimum value function and we will derive the modified global duality theorem without the assumption (AX). Throughout this section (A1)$\sim$(A4) are assumed as usual.

For $\gamma \geq 0$, $y \in R^m$, $y_J \in R^{|J|}$ we define

$$w^\gamma(y): = w(y) - \gamma||y-b||^2$$

$$w^\gamma_J(y_J): = w_J(y_J) - \gamma||y_J-b_J||^2$$

where $||y-b||^2 = \sum_1^m (y_j-b_j)^2$ and $||y_J-b_J||^2 = \sum_J (y_j-b_j)^2$.

Let T^γ : $R^{m+1} \longrightarrow R^{m+1}$ be defined by $T^\gamma(y_o,y): = (y_o-\gamma||y-b||^2, y)$ for $\gamma \geq 0$. It is easily verfied that T^γ gives a homeomorphism of R^{m+1} and maps

$\{(y_o,y) \mid w(y) \geq y_o\}$ homeomorphically onto $\{(y_o,y) \mid w^\gamma(y) \geq y_o\}$. Let us define

$$\mathrm{eff}^\gamma(K) := T^\gamma(\mathrm{eff}(K))$$

$$\mathrm{eff}^\gamma(X) := \{x \in R^n \mid (f(x),g(x)) \in \mathrm{eff}^\gamma(K)\} .$$

Since T^γ is continuous $\mathrm{eff}^\gamma(K)$ is compact by (A3).

Let us consider a program

$$(9.1) \qquad \max_{R_{++}^{|J_o|}} \{F_q(y_o,y_J) \text{ subject to } w_J^\gamma(y_J) \geq y_o\}$$

for some $q \geq -1$ and $\gamma \geq 0$. Then we have

<u>Lemma</u> 9.2 (local support)

<u>In the program</u> (9.1), <u>for sufficiently large</u> γ, <u>there exists a neighborhood</u> U^γ <u>of</u> (y_o^*,y_J^*) <u>in</u> $R_{++}^{|J_o|}$ <u>such that for any</u> $q \geq -1$ <u>and any</u> $(y_o,y_J) \in U^\gamma \cap \{(y_o,y_J) \mid w_J^\gamma(y_J) \geq y_o\}$, <u>we have that</u>

$$F_q(y_o,y_J) \leq F_q(y_o^*,y_J^*)$$

where equality holds only if $(y_o,y_J) = (y_o^*,y_J^*)$

<u>proof</u> For computational convenience, we consider a program

$$(9.3) \qquad \max_{R_{++}^{|J_o|}} \{G_q(y_o,y_J) \text{ subject to } w_J^\gamma(y_J) \geq y_o\}$$

where $\qquad G_q(y_o,y_J) = \log F_q(y_o,y_J)$ as before.

Since we have $\nabla w_J^\gamma(b_J) = \nabla w_J(b_J)$, the entire proof of Proposition 4.5 is applied. So we obtain that (y_o^*,y_J^*) satisfies the Kuhn-Tucker conditions with the Lagrange

multiplier $v^* = (u^{*T}b)^{-1}$ which does not depend on γ and q. Note that $\nabla^2 w_J^\gamma(b_J)$ is negative definite for sufficiently large γ, because $\nabla^2 w_J^\gamma(b_J) = \nabla^2 w_J(b_J) - 2\gamma I_{|J|}$. Therefore, by Remark 4.8 (y_o^*, y_J^*) satisfies the second order sufficient conditions for (9.3) for sufficiently large γ. Hence by Proposition 8.2 we complete the proof.

QED

Lemma 9.4 (global support)

Let $\gamma \geq 0$ *be sufficiently large such that* $\nabla^2 w_J^\gamma(b_J)$ *is negative definite.* Then for sufficiently large q and for any $(y_o, y) \in \mathrm{eff}^\gamma(K)$, *we have*

$$F_q(y_o, y) \leq F_q(y_o^*, y^*)$$

where equality holds only if $(y_o, y_J) = (y_o^*, y_J^*)$.

proof

We separate this proof into two part: a neighborhood V^γ of (y_o^*, y^*) and outside this neighborhood $\mathrm{eff}^\gamma(K) - V^\gamma$. The proof of the last part is exactly the same as the one in Theorem 5.1. So we will prove the first part. Let us define

$$V^\gamma := (U^\gamma \times R_{++}^{|I \backslash J_o|}) \cap \{(y_o, y) \mid w^\gamma(y) \geq y_o\} .$$

$(y_o^1, y^1) \in V^\gamma \Longrightarrow w_J^\gamma(y_J^1) = w_J(y_J^1) - \gamma||y_J^1 - b_J||^2 \geq w(y^1) - \gamma||y^1 - b||^2$
$= w^\gamma(y^1) \geq y_o^1$ because $w_J(y_J^1) \geq w(y^1)$ and $||y^1 - b||^2 \geq ||y_J^1 - b_J||^2$.
$\Longrightarrow (y_o^1, y_J^1) \in U^\gamma \cap \{(y_o, y_J) \mid w_J^\gamma(y_J) \geq y_o\} \Longrightarrow F_q(y_o^1, y_J^1) \leq F_q(y_o^*, y_J^*) \Longrightarrow$
$F_q(y_o^1, y^1) \leq F_q(y_o^*, y^*)$ for $q \geq -1$ and equality holds only if $(y_o^1, y_J^1) = (y_o^*, y_J^*)$.

QED

<u>Theorem</u> 9.5

<u>Let</u> $\gamma \geq 0$ <u>be sufficiently large such that</u> $\nabla^2 w_J^\gamma(b_J)$ <u>is negative definite</u>. Then under the assumptions (A1) - (A4), for sufficiently large q we have

$$f(\underset{\sim}{x}^*) = \min_{u \geq o} \max_{\mathrm{eff}^\gamma(X)} \{f(\underset{\sim}{x}) + u^T(g(\underset{\sim}{x}) - \underset{\sim}{b})\}$$

<u>proof</u>

If we replace eff(K) and $\mathrm{eff}(\underset{\sim}{K})$, respectively, by $\mathrm{eff}^\gamma(K)$ and $\mathrm{eff}^\gamma(K)$, then the proof of this theorem follows exactly from the proof of Theorem 7.4.

<u>QED</u>

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The canonical dual formulation (the Lagrangian function) is considered. No concavity is assumed. The existence of the nonlinear support to the hypograph of the optimum value function is proved under certain assumptions. By the transformation, this nonlinear support becomes linear and the global duality is derived.

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